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Hybrid Bases in Graphs

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Abstract

In this paper we introduce a new concept, that of hybrid base, which is a maximal circuitless and cutsetless subset of a graph. Although this concept of simultaneous circuitlessness and cutsetlessness has been used in proofs of some theorems in so called hybrid graph theory, it has not received much attention. Only largest circuitless and cutsetless subsets (hybrid bases of maximum cardinality) have been recognised as important and then only as an auxiliary notion. In contrast to maximally circuitless subsets (trees) or to maximally cutsetless subsets (cotrees), hybrid bases are not of the same cardinality. This fact, although seemingly an "imperfection", is the cause of rich structure which we describe in this paper through several propositions. The concept of hybrid bases is related to several important notions in hybrid orientated graph theory. For example, it is related to maximally distant pairs of trees, to complementary pairs of trees, to perfect pairs of trees and to topological degree of freedom. It is also closely related to the problem of finding the minimum number of independent variables in the hybrid analysis of electrical networks.

§1 Introduction

We introduce a new concept, that of hybrid base, which we believe has relevance in current graph theory and within certain important applications.

A subset b of edges of a graph is said to be a **hybrid base** of the graph if it is both circuitless and cutsetless and maximal in the sense that no other circuitless and cutsetless subsets of the graph contain b as a proper subset. For example, figure 1 shows two copies of the same graph with two different hybrid bases indicated with bold lines. Any subset of a hybrid base is also both circuitless and cutsetless and will be called a **double independent subset** of the graph. It is obvious that a hybrid base or any of its subsets may be seen as part of a tree and part of a cotree at the same time, and consequently has the properties of both. But there is one important distinction: in contrast to maximally circuitless subsets (trees) of the graph or to maximally cutsetless subsets (cotrees) of the graph, maximally]circuitless **and** cutsetless subsets (hybrid bases) are not necessarily of the same cardinality. This is the case, for example, for the graph of figure 1. This "imperfection" however turns out to be an advantage.

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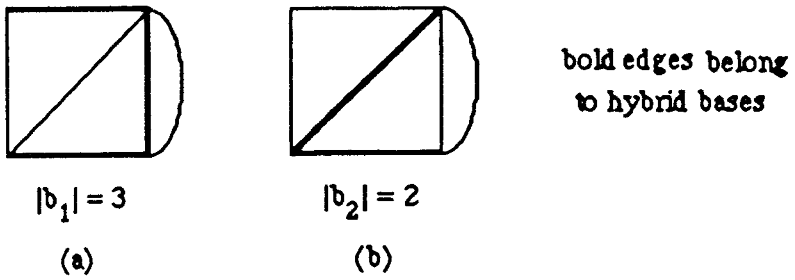


Figure 1

Although the concept of simultaneous circuitlessness and cutsetlessness has been used in proofs of some theorems [1,2,3], it is surprising that this concept has not received much attention. Only largest circuitless and cutsetless subsets have been recognised as important [3,4] but not more than as an auxiliary notion.

The part of graph theory which deals with concepts which are inherently related to both circuits and cutsets, we shall call hybrid graph theory. The promotion of new notions can be a spurious process and it may not at first be clear which are of value and whether the most useful definitions have been made. We hope that the notion of hybrid base will take its place as a notion of independent value, and will provide excellent intuitive insight within the area of hybrid graph theory. Throughout this paper we shall, without loss of generality, be concerned only with 2-connected graphs.

It was pointed out by Amari [5] that the set of edges of a graph can be divided into two distinct subsets such that the sum of the rank of one subset and the corank of the other may give a number which is less than both the rank and the corank of the graph. Not long afterwards, this number was formally defined by Tsuchiya *et al.* and called the hybrid rank [6]. In the same paper the notion of a minimal hybrid rank taken over all possible partitions of the edge set of a graph was introduced and called the toplogical degree of freedom. That paper together with the paper of Kishi and Kajitani [1] in which maximally distant pairs of trees and principal partition were introduced provide the foundation of the hybrid approach in graph theory. Since 1967 many papers have been published in this area, mostly by Japanese authors.

The concept of hybrid bases is clearly related to several important notions in hybrid orientated graph theory. For example, it is related to maximally distant pairs of trees [1], to complementary pairs of trees [3], to perfect pairs of trees [8] and to topological degree of freedom [6,7]. It is also related to the problem of finding the minimum number of independent variables in the hybrid analysis of electrical networks.

§2 Preliminaries

This section is devoted to some definitions and assertions related to material that follows. We presume that the reader is familiar with the following basic notions of graph theory: graph, edge, circuit and cutset. We take these to be primary notions that need not be defined. However we will define all other notions on the basis of these. Throughout we denote a graph by G and its edge set by E. The terms circuit, cutset, tree, cotree, forest and coforest will be used here to mean a subset of edges of a graph. A forest is a maximal circuitless subset of edges while a coforest is a maximal cutsetless subset of edges. If the graph is connected then a forest is a tree and a coforest is a cotree. In what follows, a tree will be denoted by t and a cotree by t*. The non-negative integer rank t*, related to a tree t, is called the **diameter** of the tree t. Given a tree t, any edge in the corresponding cotree t* forms exactly one circuit with edges in t. Such a circuit is called a **fundamental circuit** of G with respect to t. Similarly, any edge of the tree t defines exactly one cutset with the edges in the corresponding cotree t*. Such a cutset is called a **fundamental cutset of G** with respect to t*. If E' is a subset of E then the **rank** of E' is the cardinality of the largest circuitless subset of E', the **co-rank** of E' is the cardinality of the largest cutsetless subset of E' and the **complement** of E' is the set difference E\E' denoted by E*. By |E'| we denote the number of elements in (that is, the cardinality of) the subset of E'. Given a subsets s of a graph, we denote by $n_s(s)$ (respectively $n_c(s)$) the **maximum number of independent cutsets (circuits)** made of edges in s only. The **distance** between two (spanning) trees t_1 and t_2 of a graph, written $|t_1\backslash t_2|$, is the number of edges which are in t_1 but not in t_2. A tree t_2 is said to be **maximally distant from** another tree t_1 if $|t_1\backslash t_2| \geq |t_1\backslash t|$ for every tree t of the graph. A pair of trees (t_1,t_2) is defined to be a **perfect pair** of trees if both t_2 is maximally distant from t_1 and t_1 is maximally distant from t_2. A pair of trees (t_1,t_2) is defined to be a **maximally distant pair** of trees if $|t_1\backslash t_2| \geq |t'\backslash t''|$ for every pair (t',t"). A pair of trees (t_1,t_2) is said to be a **complementary pair** of trees if t_1 and t_2 are disjoint and their union covers the edge set E.

Assertion 1 [8]

Given a tree t_0 of a graph G, $(\forall t)\ |t_0\backslash t| \leq \text{rank } t_0^*$

Assertion 2 [8]

The following five statements are equivalent:

i) t_2 is maximally distant from t_1

ii) the fundamental circuit with respect to t_2 defined by an edge in $t_1^*\cap t_2^*$ contains no edges in $t_1\cap t_2$.

iii) the fundamental cutset with respect to t_2^* defined by an edge in $t_1\cap t_2$ contains no edges in $t_1^*\cap t_2^*$.

iv) $|t_1\backslash t_2| = \text{rank } t_1^*$

v) $|t_1\cap t_2| = n_s(t_1)$

Assertion 3 [8]

The following five statements are equivalent:

i) (t_1,t_2) is a perfect pair

ii) fundamental circuits with respect to t_1 and t_2 defined by edges in $t_1^*\cap t_2^*$ contains no edges in $t_1\cap t_2$.

iii) fundamental cutsets with respect to $t_1{}^*$ and $t_2{}^*$ defined by edges in $t_1 \cap t_2$ contains no edges in $t_1{}^* \cap t_2{}^*$.

iv) rank $t_1{}^* = |t_1 \setminus t_2| = |t_2 \setminus t_1| =$ rank $t_2{}^*$

v) $n_s(t_1) = |t_1 \cap t_2| = n_s(t_2)$

In proving theorems we shall occasionally refer to the following three theorems which one can found in the graph theoretic literature, for example in [10,11].

Orthogonality theorem

Given a graph G let C be a circuit and S be a cutset of G. Then $|C \cap S| \neq 1$.

Painting theorem (also known as the **Colouring theorem**)

Given a graph G let $\{e\}$, E_1 and E_2 form a partition of the edge set E of G. Then either e forms a circuit with edges in E_1 only or a cutset with edges in E_2 only, but not both.

Maximal independence theorem

Let A be any edge subset of a graph G. Then all maximal circuitless (cutsetless) subsets of A are of the same cardinality.

A more general version of the Painting theorem, together with a large number of its corollaries can be found in [12] .

§3 Double independence and hybrid base

A subset of edges of a graph G is said to be a **double independent** subset of E if it contains no circuits and no cutsets of the graph G. This concept, although it has never been given a specific name has been used in proofs of some theorems [1,2,3,4].

A double independent subset d of edges of a graph G is said to be a **hybrid base** of G if it is maximal in the sense that no other double independent subset of G contain d as a proper suset. In other words a double independent subset d of G is a hybrid base iff for an arbitrary edge e in d*, $d \cup \{e\}$ is not a double independent subset of G. Obviously, an edge subset of G is a double independent subset of G iff it is a subset of a hybrid base of G.

In contrast to maximally circuitless subsets (trees) or to maximally cutsetless subsets (cotrees), maximally circuitless and cutsetless subsets (hybrid bases) need not have the same cardinality. As illustrations we present the following examples:

Example 1

For the graph shown in figure 2 all hybrid bases are listed below and classified by cardinality into two groups:

a) with cardinality 2:

(3,6)

b) with cardinality 3:

(1, 3, 4)	(1, 3, 5)	(1, 4, 6)
(1, 5, 6)	(2, 3, 6)	(2, 3, 5)
(2, 4, 6)	(2, 5, 6)	

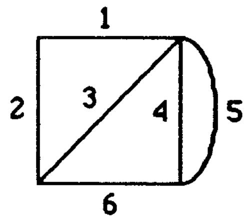


Figure 2

This is one of the smallest examples with at least two different cardinalities of hybrid bases.

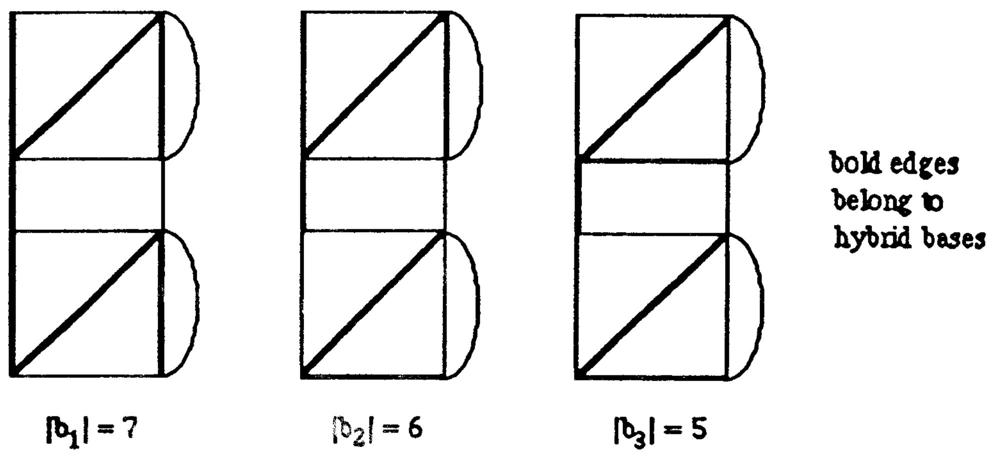


Figure 3

Example 2

We employ an augmented version of the graph of figure 2, shown in figure 3. All hybrid bases are classified by cardinality in three groups: with cardinality 5, with cardinality 6 and with cardinality 7. One representative for each group is shown in figure 3. The list of all hybrid bases for the graph is shown in Appendix 1.

Remark 1

Starting with the graph of example 1, it is possible to build a class of graphs by consecutively adding new 'floors' which are copies of the same graph . Adding one floor above the ground floor gives the graph of figure 3 for which there are three different cardinalities of hybrid bases: 5,6 and 7. The result of building n-1 floors over the ground floor, is shown in figure 4.

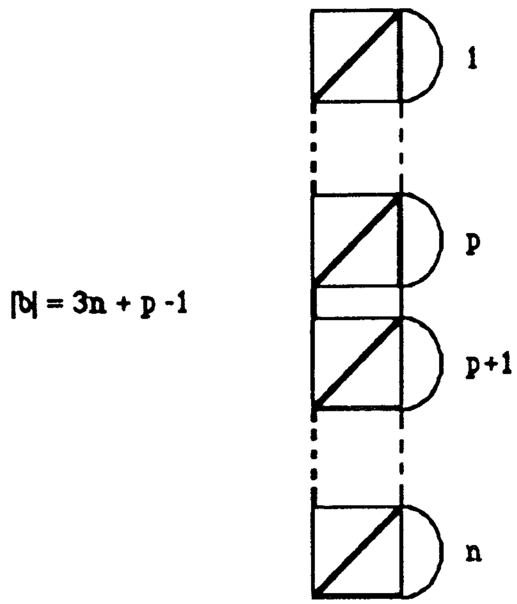


Figure 4

It is not difficult to prove that all hybrid bases of the graph obtained (which has n floors including the ground floor) can be classified by cardinality into n+1 groups represented by cardinalities 3n-1, 3n , ..., 4n-1.

We now present some properties of double independent subsets and hybrid bases based on the Painting theorem. The first proposition gives a necessary and sufficient condition for a subset of edges to be a double independent subset.

Proposition 1

A subset d of edges is a double independent subset of a graph iff each edge in d forms both a circuit and a cutset with edges in d* only.

Proof

Let an edge e belongs to a subset d. Then according to Painting theorem applied on triple ($d\setminus\{e\}$, $\{e\}$, d*) e does not form either a circuit or a cutset with edges in d only iff e forms both a circuit and a cutset with edges in d* only. □

Next proposition is a strengthen version of Proposition 1.

Proposition 2

Let d be a double independent subset of a graph and let $e \in d^*$. Then $d \cup \{e\}$ is also double independent iff e belongs to a circuit and to a cutset made of edges in d* only.

Proof

According to the Painting theorem applied to the triple (d, {e}, $d^*\setminus\{e\}$), e belongs to both a circuit and a cutset made of edges in d* only iff e does not form either a circuit or a cutset with edges in d only. Because d is double independent, the last statement means that $d \cup \{e\}$ is also double independent. □

Note

In proving Proposition 1, Painting theorem is applied to the triple ($d\setminus\{e\}$, $\{e\}$, d^*), whereas in the case of Proposition 2 this theorem is applied to the triple (d, $\{e\}$, $d^*\setminus\{e\}$). The difference is in the choice of the edge e.

Now we establish necesary and sufficient conditions for a double independent subset to be a hybrid base.

Proposition 3

Given a double independent subset d of a graph the following statements are equivalent

i) d is a hybrid base

ii) any circuit and any cutset made of edges in d^* only are disjoint

iii) any edge in d^* forms a circuit and/or a cutset with edges in d only

Proof

i) $\Leftrightarrow$ ii) A double independent subset d is a hybrid base if for each $e\in d^*$, $d\cup\{e\}$ is not double independent. But according to Proposition 2 it is true iff there is no edge that belongs at the same time to a circuit and a cutset made of edges in d^* only, that is, iff any circuit and any cutset made of edges in d^* only, are disjoint.

ii)$\Leftrightarrow$iii) The statement ii) is true iff there is no edge in d^* that forms both a circuit and a cutset with edges in d^* only, that is, iff for each edge $e\in d^*$, e does not form a circuit and/or a cutset with edges in d^* only. By applying the Painting theorem to the triple (d, $\{e\}$, $d^*\setminus\{e\}$), we deduce that e forms a cutset and/or a circuit with edges in d only. □

The following algorithm for finding a hybrid base is based on Proposition 3.

Algorithm 1

(To find a hybrid base b)

Input: A graph G.

begin

f ← set of all edges of the graph G

b ← ∅

while f is nonempty **do**

    begin

    Contract the edges in G that belong to b and in the graph obtained identify edges that belong to self circuits. Denote the set of these edges by C.

    Remove the edges in G that belong to b and in the graph obtained identify edges that belong to self cutsets. Denote the set of these edges by S.

    f ← the complement of the union $b \cup C \cup S$

    Choose any edge $e\in f$

    $b \leftarrow b\cup\{e\}$

    end

end of Algorithm

According to Proposition 3, each edge in the complement of a hybrid base b, forms a circuit or a cutset with edges in b only. Because a hybrid base is always part of a tree, a circuit that an edge in the complement of b forms with edges in b only is a fundamental circuit with respect to the tree and consequently is unique. From dual reasoning, a cutset that an edge in the complement forms with edges in b only is also unique. But, nevertheless, an edge from the complement of b may form at the same time both a circuit and a cutset. For example, in the graph of figure 5, the marked edges which belong to the complement of a hybrid base (formed from the bold edges) make both a circuit and a cutset with the hybrid base.

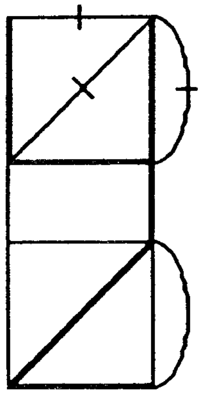


Figure 5

Thus, the edges in d* can be classified into three groups: edges that do not form cutsets with edges in d only, edges that do not form circuits with edges in d only and edges that form both a circuit and a cutset with edges in d only. According to the Painting theorem an edge belongs to the first (respectively, second) group iff it belongs to a circuit (a cutset) made of edges in d* only. According to the same theorem an edge belongs to the third group iff it does not belong to either a circuit or a cutset made of edges in d* only.

§4 Rank considerations

Recall n_s and n_c. Given a double independent subset d of a graph G, the following proposition relates the numbers $n_s(d^*)$ and $n_c(d^*)$ to the cardinality of d and d* to the rank and the corank of d and d* and to the rank and the corank of G.

Proposition 4

Given a graph G and a hybrid base d of G, the following relations hold:

$$\text{rank } G = |d| + n_s(d^*) = \text{rank } d^* = |d^*| - n_c(d^*), \quad \text{rank } d = |d|$$

$$\text{corank } G = |d| + n_c(d^*) = \text{corank } d^* = |d^*| - n_s(d^*), \quad \text{corank } d = |d|$$

Proof

This follows the statement and proof of the following lemma.

Lemma 1

Let ρ (μ) be a rank (corank) function of a graph G with edge set E. Then, for all $A \subseteq E$, the following hold:

$$|A| = n_c(A) + \rho(A)$$

$$|A| = n_s(A) + \mu(A)$$

$$\rho(E) = n_s(A) + \rho(A^*)$$

$$\mu(E) = n_c(A) + \mu(A^*)$$

Proof

Let A_s (respectively, A_c) be a maximal subset of A which does not contain cutsets (circuits). Then any edge in $A \setminus A_s$ (respectively, $A \setminus A_c$) forms a cutset (circuit) with edges in A_s (A_c) only and consequently $|A \setminus A_s|$ ($|A \setminus A_c|$) equals a maximal number of independent cutsets (circuits) of a graph G, made of edges in A only. But:

$$|A \setminus A_s| = |A| - \mu(A)$$

$$|A \setminus A_c| = |A| - \rho(A)$$

and therefore we obtain

$$n_s(A) = |A| - \mu(A) \qquad (1a)$$

$$n_c(A) = |A| - \rho(A) \qquad (1b)$$

Let $\rho^B(\mu^B)$ be a rank (respectively, corank) function of contraction (reduction) of G from E to $B \subset E$. Then for all $A \subseteq E$ [11] :

$$\rho^B(A) = \rho(A \cup (E \setminus B)) - \rho(E \setminus B)$$

$$\mu^B(A) = \mu(A \cup (E \setminus B)) - \mu(E \setminus B)$$

Let A=B, then we obtain:

$$\rho^A(A) = \rho(E) - \rho(A^*) \qquad (2a)$$

$$\mu^A(A) = \mu(E) - \mu(A^*) \qquad (2b)$$

On the other hand, for ρ and μ the following relationships hold for all $A \subseteq E$, [11] :

$$|A| - \mu(A) = \rho(E) - \rho(A^*) \qquad (3a)$$

$$|A| - \rho(A) = \mu(E) - \mu(A^*) \qquad (3b)$$

Comparing (2a) with (3a) and (2b) with (3b) we obtain:

$$\rho^A(A) = |A| - \mu(A) \qquad (4a)$$

$$\mu^A(A) = |A| - \rho(A) \qquad (4b)$$

Comparing (1a) with (4a) and (3a) and comparing (1b) with (4b) and (3b) we conclude that the following relations hold:

$$\rho(E) = n_s(A) + \rho(A^*) \tag{5a}$$

$$\mu(E) = n_c(A) + \mu(A^*) \tag{5b}$$

□

Proof (of Proposition 4)

Recall relations (1a,b) and (5a,b) and specify A respectively as:

i) $A = d$

ii) $A = d^*$

Obviously: $\rho(E) = \text{rank } G$, $\mu(E) = \text{corank } G$

$n_s(d) = 0$, $n_c(d) = 0$

$\rho(d) = |d|$, $\mu(d) = |d|$.

and therefore we finally obtain:

i) rank $G = \rho(d^*)$, $|d| = \text{corank } d$

corank $= \mu(d^*)$, $|d| = \text{rank } d$

and ii) rank $G = n_s(d^*)+|d|$, rank $d^* = |d^*| - n_c(d^*)$

corank $G = n_c(d^*)+|d|$, corank $d^* = |d^*| - n_s(d^*)$ □

Corollary 1 (to Proposition 4)

The sums $|d| + n_c(d^*)$ and $|d| + n_s(d^*)$ are invariants of a graph.

Corollary 2 (to Proposition 4)

The numbers $n_c(d^*)$ and $n_s(d^*)$ are invariants of the set of all double independent subsets of the same cardinality.

Note: to each class of double independent subsets we may associate a triple $(d \mid n_c(d^*), n_s(d^*))$. If d_1 and d_2 are two double independent subsets of a graph and $|d_1| = |d_2| + k$ then according to Proposition 4, $n_c(d_2^*) = n_c(d_1^*) - k$, and $n_s(d_2^*) = n_s(d_1^*) - k$.

Example 3 (Illustrating Corollaries 1 and 2)

Two copies of the same graph with two different double independent subsets of the same class, are shown in figure 6. Note that d_2 is a hybrid base while d_1 is not. Nevertheless the statements of Corollaries 1 and 2 hold. Both d_1 and d_2 are characterised by the triple $(2 \mid 1, 2)$.

Proposition 5

A subset b of edges of a graph G is a hybrid base of G iff b^* is a minimal subset with the properties rank b^*= rank G and corank b^*=corank G in the sense that no other subset of b^* has the same property.

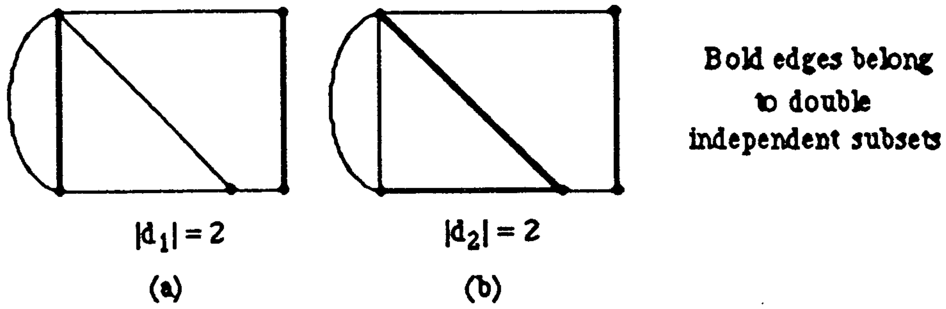


Figure 6

The following algorithm for finding a hybrid base of a graph is based on Proposition 5.

Algorithm 2
(To find a hybrid base b)
Input: A graph G
begin
G'$\leftarrow$ G
G"$\leftarrow$ G
b $\leftarrow \emptyset$
while corank G'=corank G and rank G"=rank G **do**
  begin
  Choose an edge $e \in b^*$ that does not form either a selfcircuit of G' or a selfcutset of G".
  Contract e in G' and remove e from G". Denote the graphs obtained by G_e' and G_e'' respectively.
  G' $\leftarrow G_e'$
  G" $\leftarrow G_e''$
  b $\leftarrow$ b $\cup$ {e}
end
end of Algorithm

§5 Hybrid bases that belong to a prescribed tree

Unlike Algorithms 1 and 2 in which a hybrid base is obtained by consecutive augmentation of a double independent subset, here we search for hybrid bases by making consecutive reductions of a set of edges that contains a hybrid base as a proper subset. Evidently, any hybrid base is a subset of a tree and so it is convinient to start with a circuitless subset that contains a hybrid base. The dual proposition can be formulated with cutsetless subsets. The main advantage in searching for hybrid bases by making reductions of trees is in the following: it is possible to predict the cardinality of a hybrid base that belongs to a prescribed tree, before we have actually found a hybrid base. Moreover, it is possible to find a hybrid base with prescribed cardinality by choosing a tree with appropriate diameter.

Proposition 6

Let g be a circuitless subset of edges of a graph having the following property:

(p) each edge in g^* forms a circuit or a cutset with edges in g only.

Then for any edge $e \in g$, the subset $g_1 = g \backslash \{e\}$ has property (p) iff the conjunction of the following two statements holds:

i) e belongs to to a cutset made of edges in g only, and

ii) any cutset that e forms with edges in g^* and any circuit made of edges in g^* only, are disjoint.

Note. Since g is circuitless, the edge e does not form a circuit with edges in $g \backslash \{e\}$ only and therefore, according to the Painting theorem, e forms at least one cutset with edges in g^* only.

Before we prove Proposition 6, we present the following lemma.

Lemma 2

Given an arbitrary subset g of a graph, suppose that $e \in g$ forms a cutset S_e with edges in g^* only and that $x \in g^*$ forms a circuit C_x with edges in g only, then $e \in C_x$ iff $x \in S_e$.

Proof

The subsets $S_e \backslash \{e\} \subseteq g^*$ and $C_x \backslash \{x\} \subseteq g$ are obviously disjoint and hence $\{e\} \subset S_e \cap C_x \subseteq \{x, e\}$. Since $|S_e \cap C_x| \neq 1$, we conclude that $S_e \cap C_x = \{x, e\}$, that is $e \in C_x$ iff $x \in S_e$. □

Proof (of Proposition 6)

$\Rightarrow$ Suppose that $g_1 = g \backslash \{e\}$ has property (p), then $e \in g_1^*$ forms a circuit or a cutset with edges in g_1 only. Because g is circuitless, e does not form a circuit with edges in g_1 only. This leads to the following two conclusions:

a) e forms a cutset with edges in $g_1 = g \backslash \{e\}$ only, and

b) e forms a cutset with edges in g^* only (according to the Painting theorem applied to the triple (g_1, $\{e\}$, g^*)). Obviously a) is equivalent to i). Suppose now that condition ii) is not true. That is, suppose that there exists an edge $x \in S_e \backslash \{e\} \subset g^*$ that forms a circuit with, say C_x, with edges in g^* only. Then according to the Painting theorem, x does not form a cutset with edges in g only and, consequently, neither with edges in $g \backslash \{e\}$ only. According to property (p) of g, x must form a circuit with edges in g only. Denote this circuit by C_x. Since g is circuitless, C_x is the unique circuit that x forms with in g. Applying Lemma 2 to the pair (S_e,C_x), we conclude that $e \in C_x$. Now, if we remove the edge e from g and add it to g^*, this unique circuit is destroyed. Thus x both does not form a cutset and does not form a circuit with edges in $g \backslash \{e\}$ only. This contradicts the assumption that $g \backslash \{e\}$ has property (p). Hence, condition ii) holds.

$\Leftarrow$ Suppose that the subset $g_1 = g \backslash \{e\}$ does not have property (p). Then there exists an edge $x \in g_1^*$ that does not form either a circuit or a cutset with edges in g_1 only. Because g contains e and g is circuitless, then e does not form a circuit with edges in $g \backslash \{e\} = g_1$ only.

Case 1. Suppose also that e does not form a cutset with edges in g_1. Then condition i) does not hold and consequently the conjuction of i) and ii) is not true.

Case 2. Suppose that e forms a cutset with edges in g_1 only. Then $x \neq e$. Because of the assumptions that g has property (p) and g_1 does not, the edge x forms either a circuit or a cutset with edges in g only such that either is destroyed by removing e from g. We shall first show that if x would form a cutset with edges in g only then this cutset cannot be destroyed by removing e from g. Actually, if S_α is a cutset that x forms with edges in g only, then either S_α contains e or not. If S_α does not contain e then, after removing e, x still forms a cutset with edges in g_1. If S_α contains e then because e forms a cutset with S_e with edges in g_1 only, $(S_x \cup S_e) \setminus (S_\alpha \cup C_x)$ is still a cutset that contains edges x and edges in g_1 only. Hence the assumption that x forms a cutset with edges in g only contradicts the assumption that g_1 does not have property (p).

At the begining of Case 2, we claimed that the edge x forms only a circuit C_x with edges in g only. Due to the assumption that g is circuitless, C_x is the unique circuit x forms with edges in g only. Therefore, C_x can be destroyed iff e belongs to C_x. Thus $e \in C_x$ and applying Lemma 2 we conclude that x must belong to $S_e \setminus \{e\}$.

On the other hand, according to the Painting theorem, if x does not form a cutset with edges in g_1 only, then x forms a circuit C_x^* with edges in g_1^*. Also, applying the Painting theorem, we conclude that e does not belong to C_x^* because otherwise e cannot form a cutset with edges in g_1 only. As we pointed out in the note, e forms at least one cutset with edges in g^* only. Thus we finally have the following: both a cutset that e forms with edges in g^* and a circuit that made of edges in g^* only (that is, C_x^*) contain the edge e and therefore are not disjoint. This contradicts condition ii) and consequently the conjuction of i) and ii). □

Proposition 6 enables us to develop a procedure for checking whether a tree contains a hybrid base as a proper subset. The procedure consists of consecutive applications of the following two steps.

a) Given a circuitless subset g, find an edge $e \in g$ such that both conditions i) and ii) of proposition 6 are fulfilled.

b) $g \leftarrow g \setminus \{e\}$

The input set can be a tree of the graph. The procedure terminates when either there are no more cutsets of the graph in g or when there are such cutsets but none of the edges that belong to them satisfies condition ii). There are three possibilities:

1). The procedure cannot start because there are no edges for which conditions i) and ii) are simultaneously satisfied. For example this is the case with the graph of figure 7a.

2) The procedure can start but in the final result we obtain a subset which is not double independent (figure 7b).

3) The procedure terminates with a double independent subset (figure 7c).

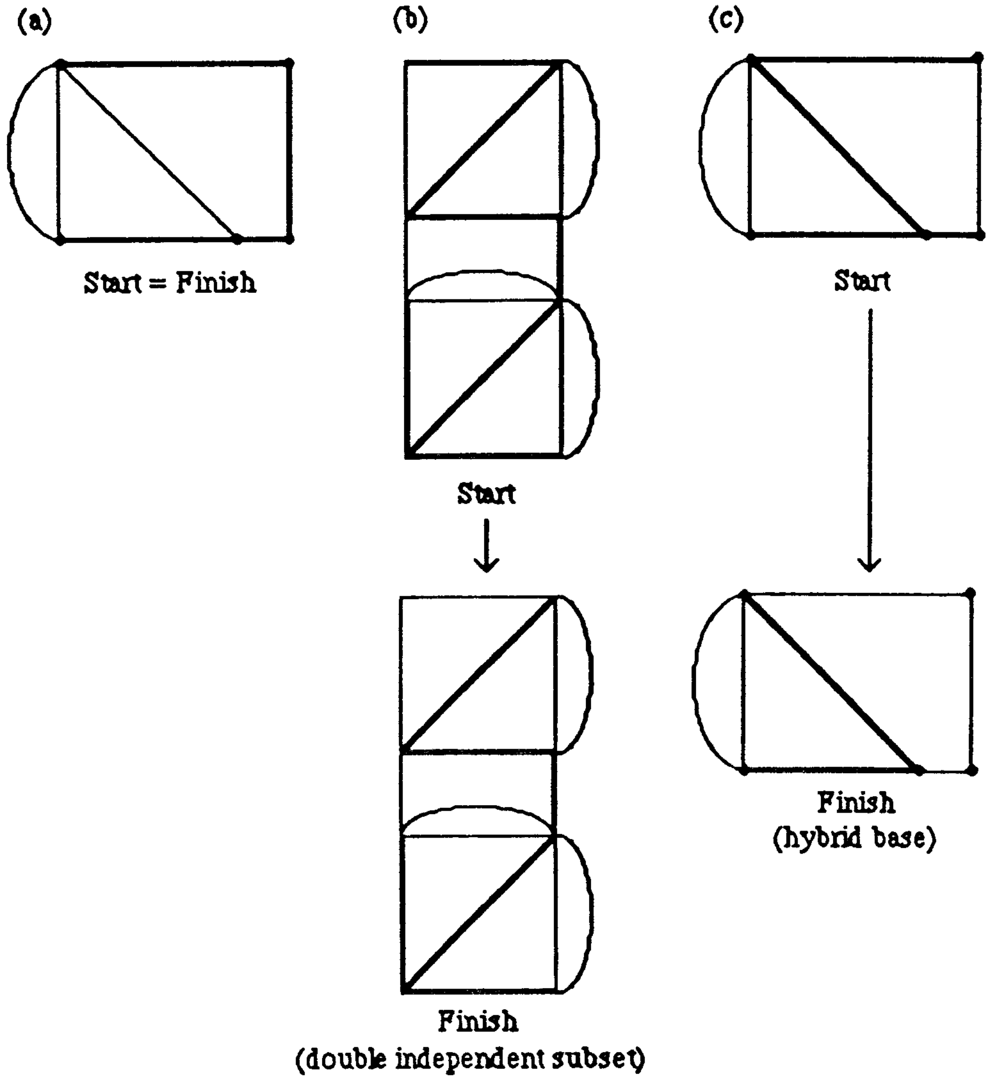


Figure 7

A tree may contain more than one hybrid base as in the example of fig 8 where there are 12 distinct hybrid bases (only 3 are indicated).

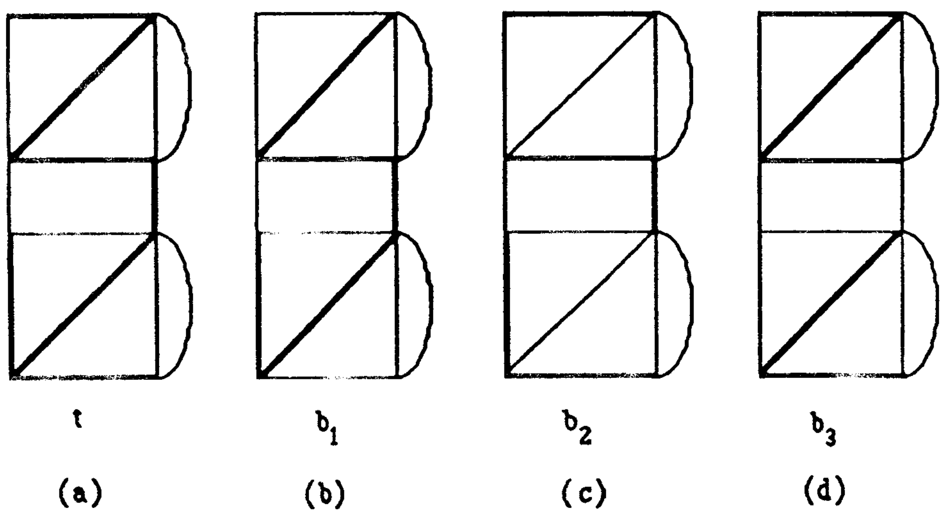


Figure 8

The above procedure is formalized in the following algorithm.

Algorithm 3

(To find all hybrid bases that belong to a prescribed tree)

Input :A graph G and a tree t of G

begin

$H \leftarrow \emptyset$

Remove the edges in G that belong to t and in the graph obtained find the set of edges that do not form selfcutsets. Denote this set by C.

Contract the edges in G that belong to t* and find the set of all trees in the graph obtained. Denote this set of trees by T.

while T is nonempty **do**

begin

Choose any tree $\tau \in T$.

Find the union of all fundamental cutsets of G defined by edges in τ.

Denote this union by $S(\tau)$.

if $C \cap S(\tau)$ is nonempty **then** $\{t \setminus \tau$ is a hybrid base$\}$ $H \leftarrow H \cup \{t \setminus \tau\}$

else $\{t \setminus \tau$ is not a hybrid base$\}$ $H \leftarrow H$

$T \leftarrow T \setminus \{\tau\}$

end

if H is not empty **then** output "H is a set of all hybrid bases that belong to t"

else output "there are no hybrid bases that belong to t"

end of Algorithm.

Proposition 7

Let b be a hybrid base and let t be a tree of a graph . If $b \subsetneq t$ then $|b| = \text{diameter } t = |t| - n_s(t)$.

Proof

Given a graph G and a tree t, let b be a hybrid base that belong to t as a proper subset. Then according to Proposition 3 and the fact that b belong to t, any edge in $t \setminus b$ forms a cutset with edges in b only. On the other hand b itself is cutsetless and therefore b is a maximal cutsetless subset of t. Thus each edge in $t \setminus b$ forms a cutset with edges in b only and hence $n_s(t) = |t \setminus b|$. But $|t \setminus b| = |t| - |b|$ and therefore $n_s(t) = |t| - |b|$. According to Lemma 1 applied to A=t, rank $t^* + n_s(t) = |t|$. From the last two relations we obtain $|b| = \text{rank } t^*$ (= diameter t) $= |t| - n_s(t)$. □

As an immediate consequence of Proposition 7 we have the following corollary:

Corollary 3 (to Proposition 7)

All hybrid bases that belong to the same tree of a graph are of the same cardinality.

According to Proposition 7, to find a hybrid base of prescribed cardinality c, we have to apply Algorithm 3 on a tree of diameter equal to c.*The minimal cardinality in the set of all hybrid bases of a graph is obviously not less then the minimal diameter in the set of all trees of that graph.*

§6 Hybrid bases and perfect pairs of trees

The notions of double independence and hybrid base are closely related to pairs of trees, particulary to perfect pairs of trees.

The first proposition in this section relates double independent subsets of a graph and pairs of trees. The same result was implicitly employed in [1] in the proof of Theorem 11.

Proposition 8

A subset of edges of a graph is a double independent subset iff it can be represented as a set difference of a pair of trees of the graph.

Proof

$\Rightarrow$ Let d be a double independent subset of a graph G and let t be a tree that contain d as a proper subset. Because no any circuit of G made of edges in d only, the graph G_d obtained from G by removing all edges that belong to d, has the same rank as G. Every circuitless subset of G_d is a circuitless subset of G too. Hence, a tree t' of G_d that includes $t\backslash d$ as a proper subset is also a tree of G with the same property. Consequently, (t,t') is a pair of trees for which $t\backslash t' = d$.

$\Leftarrow$ Let (t_1,t_2) be a pair of trees of a graph G. Then, $t_1\backslash t_2$ is a subset of both t_1 and t_2^*. Therefore, $t_1\backslash t_2$ is a double independent subset of G. □

Because each hybrid bases of a graph is double independent we conclude from Proposition 8 that any hybrid base is a set difference of a pair of trees. The converse statement is not generally true. For example, figure 9 shows six copies of a graph and for each copy a different subset of edges is indicated by the use of bold edges. The subset of edges d_1 is a double independent subset and so is d_2. However, d_1 is not a hybrid base whereas d_2 is. Notice also that $d_1=t_1\backslash t_2$ and that $d_2=t_3\backslash t_4$, where t_1, t_2, t_3 and t_4 are all trees of the graph G.

The next proposition provides a link between hybrid bases and perfect pairs of trees.

Proposition 9

Let b be a hybrid base of a graph. Then there exists a perfect pair (t_1,t_2) such that $b = t_1\backslash t_2$.

Proof

Because b is a double independent subset of the graph it follows from Proposition 8 that there exists a pair of trees (t_1,t_2) such that $b = t_1\backslash t_2$. Thus $t_1\backslash t_2$ is a hybrid base of G and, according part

iii) of Proposition 3, each edge of its complement (which includes $t_1{}^*\cap t_2{}^*$) makes a circuit or/and a cutset with the elements of $t_1\backslash t_2$ only. But $t_1\backslash t_2$ together with $t_1{}^*\cap t_2{}^*$ belongs to $t_2{}^*$ and hence the edges in $t_1{}^*\cap t_2{}^*$ cannot make cutsets with the edges $t_1\backslash t_2$ only. Therefore edges in $t_1{}^*\cap t_2{}^*$ make circuits with edges in $t_1\backslash t_2$ only and consequently rank $t_2{}^* = |t_1\backslash t_2|$. This means that t_1 is maximally distant from the t_2. On the other hand, according to Assertion 1, rank $t_1{}^* \geq |t_1\backslash t_2|$. We now prove that for the case under consideration, equality must occur. That is, t_2 is also maximally distant from t_1. Suppose that this is not true. Then, according part ii) of Assertion 2, there exists an edge $e' \in t_1{}^*\cap t_2{}^*$ such that a fundamental circuit with respect to t_2 (defined by that edge) contains an edge $e \in t_1\cap t_2$. Consequently, $t'_2=(t_2\backslash e)\cup\{e'\}$ is again a tree and such that $t_1\backslash t_2 \subset t_1\backslash t'_2$. But subset $t_1\backslash t'_2$ is, according to Proposition 8, also a double independent subset that contains (as a proper subset) the hybrid base $t_1\backslash t_2$ which is a contradiction. Thus we have proved that t_1 is maximally distant from t_2 and vice versa. Hence (t_1,t_2) is a perfect pair. □

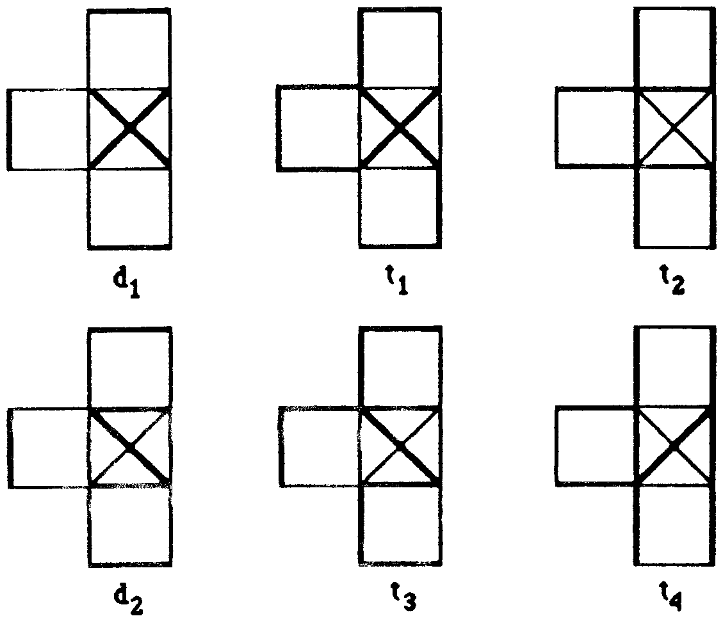


Figure 9

Remark 2

The converse of Proposition 9 is not generally true. That is, if (t_1,t_2) is a perfect pair of trees then their set difference is not necessarily a hybrid base. To see this consider figures 10 and 11. Figure 10 shows four copies of the same graph and within each a subset of edges is indicated using bold lines. Now (t_1,t_2) is a perfect pair and (by inspection) $t_1\backslash t_2$ is a hybrid base while $t_2\backslash t_1$ is not.

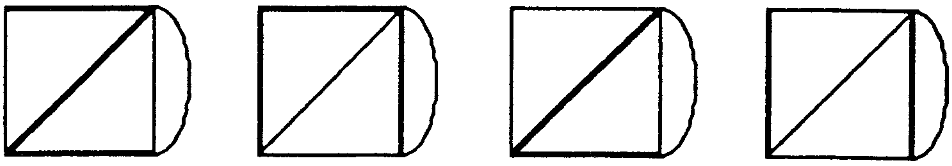


Figure 10

Figure 11 shows four copies of the same graph and again various subsets of edges are indicated using bold lines. Again (t_1,t_2) is a perfect pair while neither $t_1\backslash t_2$ nor $t_2\backslash t_1$ is a hybrid base. The marked edges (indicating $t_1\backslash t_2$ and $t_2\backslash t_1$) form neither circuits not cutsets.

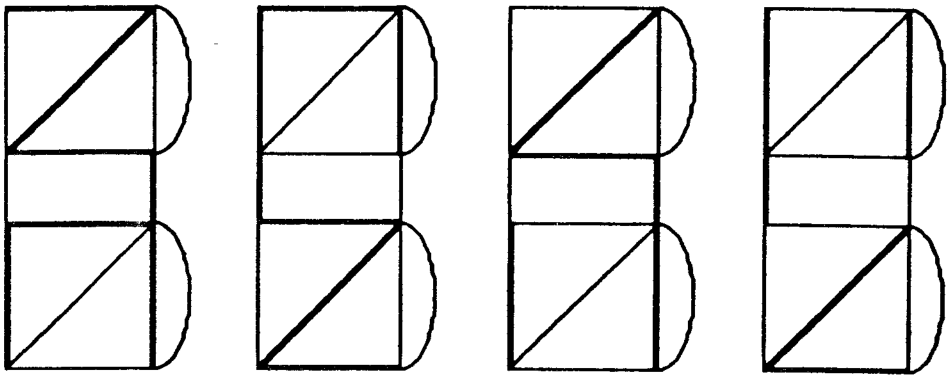


Figure 11

The next proposition gives necessary and sufficient conditions for the set difference of a hybrid pair of trees to be a hybrid base.

Proposition 10

Let (t_1,t_2) be a perfect pair of trees of a graph. Then $t_1\backslash t_2$ is a hybrid base iff each edge in $t_2\backslash t_1$ that belongs to the fundamental circuit with respect to t_2, defined by an edge in $t_1{}^*\cap t_2{}^*$, forms a circuit with edges in $t_1\backslash t_2$ only.

Proof

$\Rightarrow$ Suppose there exists an edge $e\in t_2\backslash t_1$ that belongs to the fundamental circuit with respect to t_2, defined by an edge in $t_1{}^*\cap t_2{}^*$ and at the same time e forms the fundamental circuit C_e with respect to t_1 in which at least one edge is in $t_1\cap t_2$. Thus, e does not form a circuit with edges in $t_1\backslash t_2$ only and at the same time forms a circuit C_e with edges in the complement of $t_1\backslash t_2$ only. According to the Painting theorem, applied to the triple $((t_1\backslash t_2)^*\backslash\{e\},\{e\}, t_1\backslash t_2)$, we conclude that edge e does not form a cutset with edges in $t_1\backslash t_2$ only. Since e, which is an edge in the complement of $t_1\backslash t_2$, does not form either a circuit or a cutset with edges in the complement of $t_1\backslash t_2$, according part iii) of Proposition 3, $t_1\backslash t_2$ is not a hybrid base (proof by contradiction).

$\Leftarrow$ Suppose now that each edge in $t_2 \backslash t_1$ that belongs to the fundamental circuit with respect to t_2, defined by an edge in $t_1^* \cap t_2^*$, forms a circuit with edges in $t_1 \backslash t_2$ only. The remaining edges in $t_2 \backslash t_1$ that do not belong to the circuits that edges in $t_1^* \cap t_2^*$ forms with edges in t_2, do not belong to any circuit made of edges in the complement of $t_1 \backslash t_2$. Denote the set of such edges by R. Each edge y that belong to R, according to the Painting theorem applied to the triple $((t_1 \backslash t_2)^* \backslash \{y\}, \{y\}, t_1 \backslash t_2)$, necessarily forms a cutset with edges in $t_1 \backslash t_2$ only. Thus each edge in $t_2 \backslash t_1$ forms either a circuit or a cutset with edges in $t_1 \backslash t_2$ only. On the other hand, since the pair (t_1,t_2) is a perfect pair, according to Assertion 3 all edges in $t_1^* \cap t_2^*$ form circuits with edges in $t_1 \backslash t_2$ only and all edges in $t_1 \cap t_2$ form cutsets with edges in $t_1 \backslash t_2$ only. According to part iii) of Proposition 3, $t_1 \backslash t_2$ is a hybrid base. □

As an immediate consequence of Propositions 9 and 10, we have the following proposition which gives both necessary and sufficient conditions for a subset of edges to be a hybrid base

Proposition 11

A subset $t_1 \backslash t_2$ of edges of a graph is a hybrid base iff

i) the pair of trees (t_1,t_2) is a perfect pair of trees

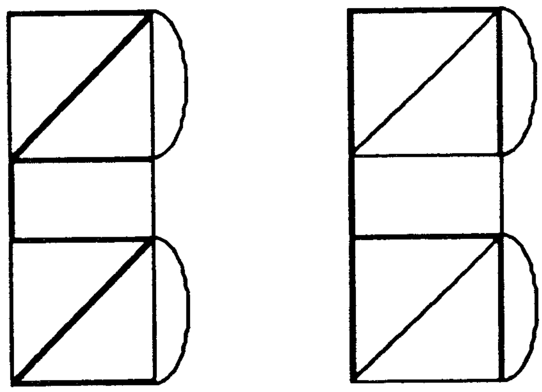
ii) each edge in $t_2 \backslash t_1$ that belongs to the fundamental circuit with respect to t_2, defined by an edge in $t_1^* \cap t_2^*$, forms a circuit with edges in $t_1 \backslash t_2$ only.

Remark 3

Condition i) and ii) of Proposition 11 are mutually independent. In other words neither (i)$\Rightarrow$(ii), nor (ii)$\Rightarrow$(i). To see that (i) does not imply (ii), it is sufficient to consider the example of figure 11 Although the pair (t_1,t_2) is a perfect pair, condition (ii) is not satisfied. Conversely, in order to see that (ii) does not imply (i), consider the example of figure 12. In this example, by inspection, condition (ii) is satisfied but rank $t_1^*=5>4=|t_1 \backslash t_2|$ and hence (t_1, t_2) is not a perfect pair.

Remark 4

Let D denote the set of all tree diameters of a graph G, let P denote the set of all perfect pair distances of G and let B denote the set of all hybrid base cardinalities associated with G. According to part iv) of Assertion 3, diameter (t_1) = distance between t_1 and t_2 = diameter (t_2) and hence P is a subset of D. On the other hand, according to Proposition 9, for the existence of a hybrid base b in a graph, it is necesary that there exists a perfect pair of trees (t_1, t_2) such that $b=t_1 \backslash t_2$. Consequently, B is a subset of P. Thus we have proved that $B \subseteq P \subseteq D$. It immediately follows that $\min D \geq \min P \geq \min B$ and $\max B \leq \max P \leq \max D$.



t_1 t_2

Figure 12

§7 Hybrid bases of maximum cardinality

The concept of hybrid bases of maximum cardinality is clearly related to several important notions in hybrid orientated graph theory. For example, it is related to maximally distant pairs of trees, to complementary pairs of trees and to topological degree of freedom. Hybrid bases of maximal cardinality (as distinct from maximum cardinality) have appeared only as an auxially notion without a specific name (except in [4] where they are called **dyads**).

Proposition 12

If (t_1,t_2) is a maximally distant pair of trees then both $t_1\backslash t_2$ and $t_2\backslash t_1$ are hybrid bases.

Proof

Suppose that one of the subsets $t_1\backslash t_2$ or $t_2\backslash t_1$ is not a hybrid base, for example the subset $t_1\backslash t_2$. Then there exists a hybrid base d that contains $t_1\backslash t_2$ as a proper subset. According to Proposition 9 there is a perfect pair of trees (t'_1,t'_2) such that $t'_1\backslash t'_2 = d$. Because $t_1 \leq d = t'_1\backslash t'_2$, we conclude that $|t_1\backslash t_2| \leq |d| = |t'_1\backslash t'_2|$ which contradicts the assumption that (t_1,t_2) is a maximally distant pair of trees. □

Corollary 4 (to Proposition 12)

The cardinality of a largest hybrid base of a graph is not less then the maximal diameter of the graph. In other words, maxB ≥ maxD.

Proof

Because a maximally distant pair is always perfect pair, the maximum tree diameter in a graph is equal to the distance between a maximally distant pair of trees. But according to Proposition 12 the set difference between a maximally distant pair of trees is a hybrid base and therefore the cardinality of a largest hybrid base is not less than the maximal diameter in the graph. □

According to Remark 4, max B ≤ max D. Thus,we have shown that the following inequalities hold simultaneously: max B ≤ max D and max B ≥ max D. Thus max B = max D. This important fact was first pointed out by Sengoku [2] and recalled by Lin [3].

Remark 5

The converse of Proposition 12 is not generally true. That is, if for a given pair of trees (t_1,t_2), both $t_1\backslash t_2$ and $t_2\backslash t_1$ are hybrid bases, then (t_1,t_2) is not necessarily a maximally distant pair of trees. In order to see this consider figure 13. This figure shows four copies of the same graph with different subsets of edges indicated with bold lines. Now $t_1\backslash t_2$ and $t_2\backslash t_1$ are both hybrid bases but (t_1,t_2) is not a maximally distant pair of trees.

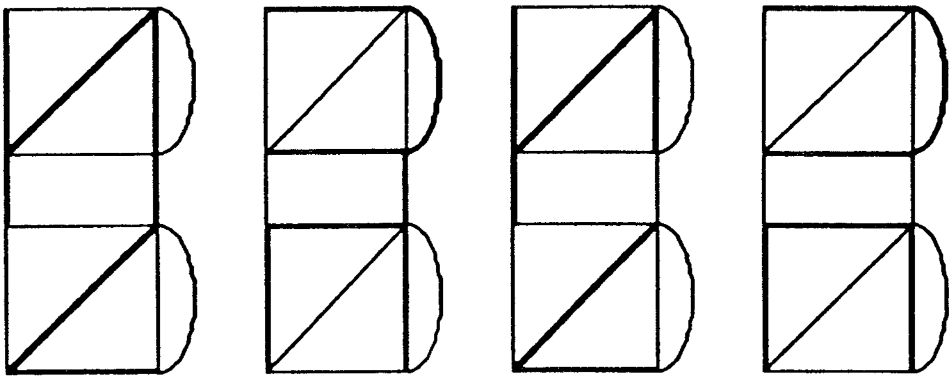


Figure 13

The next proposition is a consequence of Proposition 8. It gives necessary and sufficient conditions for a subset of the edges of a graph to be a hybrid base of maximal cardinality. The same statement appeared (without proof) as Theorem 15 in [4].

Proposition 13

A subset of edges b of a graph is a hybrid base of maximal cardinality (a dyad) iff b is a set difference of a maximally distant pair of trees.

Proof

$\Rightarrow$ Let b be a hybrid base of maximal cardinality. According to Proposition 8, there exists a pair of trees (t_1, t_2) such that $b= t_1\backslash t_2$. Suppose now that (t_1, t_2) is not a maximally distant pair of trees. Then, for a maximally distant pair (t_1', t_2'), which must exist, the set difference $t_1\backslash t_2'=b_1$ is also (according to Proposition 12) a hybrid base. But $|b_1|=|t_1\backslash t_2'|>|t_1\backslash t_2|=|b|$ which contradicts the assumption that b is a hybrid base of maximal cardinality.

$\Leftarrow$ Suppose that b is a set difference of a maximally distant pair of trees (t_1, t_2). Then according to Proposition 12, b is a hybrid base. Any other hybrid base , according to Proposition 8, also is a set difference of a pair of trees. But $|t_1\backslash t_2|\geq |t_1\backslash t_2'|$ for all t_1', t_2' in a graph and consequently b is of maximal cardinality. □

The following proposition relates hybrid bases to distinct and complementary trees of a graph.

Proposition 14

Given a graph G, let b be a hybrid base and let d_{b*} be a maximal double independent subset of b*. If $|b| = \text{rank } G$ then b and d_{b*} constitute a pair of distinct trees in G.

We prove the following lemma before proving Proposition 14.

Lemma 3

Given a hybrid base b of a graph , let d_{b*} be a maximal double independent subset of b*. Then d_{b*} and b are of the same cardinality.

Proof

Since b is a hybrid base, according to Proposition 9 there exists a perfect pair of trees (t_1,t_2) such that $b = t_1 \setminus t_2$. The subset $t_2 \setminus t_1$ obviously belongs to b*. Because $t_2 \setminus t_1 \subseteq t_1{}^* \cap t_2$, $t_2 \setminus t_1$ is also double independent. Since (t_1,t_2) is a hybrid pair of trees, any edge in $t_1{}^* \cap t_2{}^*$ forms a circuit with edges in $t_2 \setminus t_1$ only and any edge in $t_1 \cap t_2$ forms a cutset with edges in $t_2 \setminus t_1$ only. Therefore, $t_2 \setminus t_1$ is maximally double independent in b*. Since $|t_1 \setminus t_2| = |t_2 \setminus t_1|$, we conclude that $|b| = |d_{b*}|$. □

Proof (of Proposition 14)

Given a hybrid base b of a graph , let d_{b*} be a maximal double independent subset of b*. Then according to Lemma 3, d_{b*} and b have the same cardinality. But the cardinality of a double independent subset is always less then rankG and therefore d_{b*} is a hybrid base. □

As an immediate consequence of Proposition 14 we have the following corollary:

Corollary (to Proposition 14)

Let b be a hybrid base of a graph. Then the following statements are equivalent:

i) b* is a hybrid base

ii) $|b^*| = |b|$

iii) (b,b*) is a complementary pair of trees

§8 Application in hybrid network analysis

The classical approach in hybrid analysis of electrical networks deals with, so called, topologically complete sets of network variables. Such a set consists of a minimal number of voltages and currents whose values are sufficient to determine one of two variables (voltage or current) of every edge of the network, by means of Kirchhoff's laws only, providing that only one variable (either a voltage or a current) of an edge can contribute to the topologically complete set. In graph theoretic terms, this notion can be formalized by introducing a pair of disjoint subsets (τ,μ) of edges such that τ is circuitless, μ is cutsetless and each edge in the complement of $\tau\cup\mu$ either forms a circuit with edges in τ only or forms a cutset with edges in μ, only (see for example [2]). The set of voltages of the edges in τ and currents of the edges in μ then form a topologically complete set of network variables.

The cardinalities of complete sets of variables of a network may differ. Certainly, a complete set of the network with the minimal cardinality is of particular interest. As has been pointed out by Kishi and Kajitani [1], the minimal number of network variables in hybrid analysis coincides with the maximal distance between trees and also with the minimal hybrid rank of the graph associated with the network. The minimal hybrid rank of a graph was introduced by Tsuchiya *et al.* [6] and called the *topological degree of freedom* of a graph. On the other hand Sengoku [2] first observed that the cardinality of a largest simultaneously circuitless and cutsetles subset (in our terminology a largest hybrid base of maximum cardinality) is equal to the smallest number of topologically complete network variables.

Given a hybrid base b of a graph G, let τ (respectively, μ) consist of all edges in b that belong to the circuits (cutsets) of G that edges in b* form with edges in b only. Obviously, the union of τ and μ is equal to b. The voltages associated with edges in τ and currents associated with edges in μ form a hybrid set of variables in the sense that these variables determine at least one of two variables, voltage or current, for each edge in b*, by means of Kirchhoff's laws. This is clear because b is both circuitless and cutsetless and because any edge in b* forms a circuit with edges in τ only or a cutset with edges in μ only. If $\tau\cap\mu$ is empty then exactly one variable, either a voltage or a current is associated with each edge in b and therefore the set of voltages in τ and currents μ form a topologically complete set of variables. If $\tau\cap\mu$ is nonempty then we have an "almost" topologically complete set of variables because then each edge in $\tau\cap\mu$ contributes to the hybrid set of variables with two variables: voltage and current. The number of elements of such an "almost" topologically complete set of variables is equal to $|\tau\backslash\mu| + |\mu\backslash\tau| + 2\,|\tau\cap\mu|$. For any hybrid base b this number is always greater than or equal to $|b|$. Equality occurs iff τ and μ are disjoint, that is, iff b is a hybrid base of maximum cardinality.

The following algorithm gives a procedure for finding a pair (τ,μ) associated with a given hybrid base b.

Algorithm 4

(To find a pair (τ,μ) of subsets for a given hybrid base b)

Input

A graph G and a hybrid base b.

begin

for all $e\in b^*$ **do**

  begin

  find the circuit C_e that e forms with edges in b only

  if C_e exists **do** mark edges in $C_e\setminus\{e\}$ as v-edges

  end

for all $e\in b^*$ **do**

  begin

  find the cutset S_e that e forms with edges in b only

  if S_e exists, **do** mark edges in $S_e\setminus\{e\}$ as i-edges

  end

Denote by τ the set of all v-edges and denote by μ the set of all i-edges.

end of Algorithm

Proposition 15

Given a hybrid base b of a graph, let τ (respectively, μ) consist of all edges in b that belong to circuits (cutsets) that edges in b* form with edges in b only. If the cardinality of b is less than maximal then each such pair of subsets (τ,μ) has a nonempty intersection.

Proof

Suppose that the intersection of τ and μ is empty, that is, suppose τ and μ are disjoint. Then voltages of edges in τ and currents of edges in μ form a topologically complete set of variables. This is clear because any subset of b is both circuitless and cutsetless and any edge in the complement of the hybrid base b forms a circuit or a cutset with edges in b only. Since the union of τ and μ is equal to b and the intersection is nonempty, we conclude that the cardinality of b is equal to the cardinality of a topologically complete set of variables which is greater than or equal to the topological degree of freedom. Hence the cardinality of b is greater than or equal to the topological degree of freeedom (hybrid rank) of the graph . But, according to Remark 4, the cardinality of b is always less than or equal to the maximal distance between pairs of trees, that is, to the topological degree of freeedom, which is a contradiction. □

A main goal of this section is to show that the number of hybrid network variables can be made smaller than the topological degree of freedom by using hybrid bases of non-maximal cadinalities. For this purpose, we will relax the definition of the topologically complete set of variables by omitting the assumption that an edge can contribute in the hybrid set of variables with exactly one variable (either a voltage or a current). Thus we shall consider an "almost" topologically complete set of variables. Although such a situation with two independent variables associated with a network edge is physically admissible (for example in case of norator's edges in a network), normally the voltage and the current associated with an edge of the network are mutually tied up with the so-called constitutive relations. Briefly speaking the constitutive relation of a 1-port is described by a collection of pairs of signals of voltage and currents that are allowed by the 1-port. In the case when the 1-port is v-controlled or i-controlled, one of these two variables (voltage or current) can be found from the other and consequently, the total number of network variables needed to be associated with each edge is one. Thus when the total number of variables is equal to

the cardinality the union $\tau \cup \mu$, that is, to the cardinality of the hybrid base b. But $|b|$ is always less than or equal to the topological degree of freedom and therefore the total nuber of variables is less than the topoplogical degree of freedom.

As an illustration, we consider an electrical network with the six resistors whose constitutive relations are :

$v_1 = v_1(i_1)$ (CR1)

$v_2 = v_2(i_2)$ (CR2)

$i_3 = i_3(v_3)$ (CR3)

$i_4 = i_4(v_4)$ (CR4)

$i_5 = i_5(v_5)$ (CR5)

$v_6 = v_6(i_6)$ (CR6)

Let G be the graph (associated with the network under consideration) shown in figure 14. It is easy to see that (rank G)=(corank G)=(hybrank G)=3. Let b={3,6} be a hybrid base of G of nonmaximal cardinality (bold edges). The edge orientations are related to the edge currents and the edge voltages. The singly marked edges {1,2,6} indicate i-controlled resistors and the doubly marked edges {3,4,5} indicate v-controlled resistors.

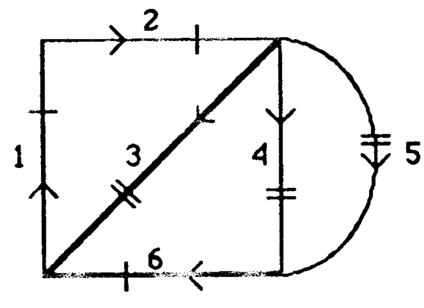


Figure 14

In this case, using Algorithm 4, we obtain $\tau = \mu = b$. Hence $\tau \cap \mu = b$ and hence $\{ v_3, i_3, v_6, i_6 \}$ constitutes a hybrid set of variables.

Doubly marked edges form circuits with edges in τ (=b) only and singly marked edges form cutsets with edges in μ (=b) only and consequently the associated Kirchhoff's laws are:

Kirchhoff's current laws (KCL)

$i_1 - i_3 + i_6 = 0$ (KCL1)

$i_2 + i_3 - i_6 = 0$ (KCL2)

Kirchhoff's voltage laws (KVL)

$- v_3 + v_4 + v_6 = 0$ (KVL1)

$- v_3 + v_5 + v_6 = 0$ (KVL2)

Substituting KCL1, KCL2, KVL1 and KVL2 into CR1, ..., CR5 and CR6, we obtain each voltage and current of the network uniquely expressed in terms of the variables v_3, i_3, v_6 and i_6:

$v_1 = v_1(i_3 - i_6)$ (CR1+KCL1)

$v_2 = v_2(-i_3 + i_6)$ (CR2+KCL2)

$i_3 = i_3(v_3)$ (CR3)

$i_4 = i_4(v_3 - v_6)$ (CR4+KVL1)

$i_5 = i_5(v_3 - v_6)$ (CR5+KVL2)

$v_6 = v_6(i_6)$ (CR6)

If we substitute (CR3) and (CR6) into the other relations, we finally obtain:

$v_1 = v_1(i_3(v_3) - i_6)$ (CR1+KCL1+CR3)

$v_2 = v_2(-i_3(v_3) + i_6)$ (CR2+KCL2+CR3)

$i_3 = i_3(v_3)$ (CR3)

$i_4 = i_4(v_3 - v_6(i_6))$ (CR4+KVL1+CR6)

$i_5 = i_5(v_3 - v_6(i_6))$ (CR5+KVL2+CR6)

$v_6 = v_6(i_6)$ (CR6)

Writing KVL for the circuit that edge 3 forms with edges 1 and 2, and KCL for the cutset that edge 6 forms with edges 4 and 5, and expressing all variables in these equations in terms of variables v_3 and i_6, according to the last equations, we finally obtain $|b|=2$ equations in terms of 2 variables $\{v_3, i_6\}$:

$v_1(i_3(v_3) - i_6) + v_2(-i_3(v_3) + i_6) + v_3 = 0$

$i_4(v_3 - v_6(i_6)) + i_5(v_3 - v_6(i_6)) - i_6 = 0$

Thus the number of equations and the number of independent variables is less then the topological degree of freedom, which we intended to show.

§10 Concluding remarks

In this paper we introduced the new concept called hybrid base of a graph. Several propositions were stated in order to closely characterise its properties and to relate this concept to some important notions in hybrid oriented graph theory. For example it is related to maximally distant pairs of trees, to perfect pairs of trees, to complementary pairs of trees, to hybrid rank and to topological degree of freedom. It is also related to the problem of finding the minimum number of independent variables in electrical networks. Also, several examples are included to help the reader gain intuitive insight. A complete list of hybrid bases of a number of graphs are shown in the appendix.

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Appendix

All Hybrid Bases for the Graph of Figure 3

a) with cardinality 5 (8 items)

1 3 6 11 14	2 3 6 11 14	3 6 7 11 14
3 6 8 11 14	3 6 9 10 12	3 6 9 10 13
3 6 9 10 14	3 6 9 11 14	

b) with cardinality 6 (80 items)

1 3 4 7 11 14	1 3 4 8 11 14	1 3 4 9 10 12
1 3 4 9 10 13	1 3 4 9 10 14	1 3 4 9 11 14
1 3 5 7 11 14	1 3 5 8 11 14	1 3 5 9 10 12
1 3 5 9 10 13	1 3 5 9 10 14	1 3 5 9 11 14
1 3 6 9 11 12	1 3 6 9 11 13	1 3 6 9 12 14
1 3 6 9 13 14	1 3 6 10 11 12	1 3 6 10 11 13
1 3 6 10 12 14	1 3 6 10 13 14	1 4 6 7 11 14
1 4 6 8 11 14	1 4 6 9 10 12	1 4 6 9 10 14
1 4 6 9 10 14	1 4 6 9 11 14	1 4 6 7 11 14
1 5 6 8 11 14	1 5 6 9 10 12	1 5 6 9 10 13
1 5 6 9 10 14	1 5 6 9 11 14	2 3 4 7 11 14
2 3 4 8 11 14	2 3 4 9 10 12	2 3 4 9 10 13
2 3 4 9 10 14	2 3 4 9 11 14	2 3 5 7 11 14
2 3 5 8 11 14	2 3 5 9 10 12	2 3 5 9 10 13
2 3 5 9 10 14	2 3 5 9 11 14	2 3 6 9 11 12
2 3 6 9 11 13	2 3 6 9 12 14	2 3 6 9 13 14
2 3 6 10 11 12	2 3 6 10 11 13	2 3 6 10 12 14
2 3 6 10 13 14	2 4 6 7 11 14	2 4 6 8 11 14
2 4 6 9 10 12	2 4 6 9 10 13	2 4 6 9 10 14
2 4 6 9 11 14	2 5 6 7 11 14	2 5 6 8 11 14
2 5 6 9 10 12	2 5 6 9 10 13	2 5 6 9 10 14
2 5 6 9 11 14	3 6 7 9 11 12	3 6 7 9 11 13
3 6 7 9 12 14	3 6 7 9 13 14	3 6 7 10 11 12
3 6 7 10 11 13	3 6 7 10 12 14	3 6 7 10 13 14
3 6 8 9 11 12	3 6 8 9 11 13	3 6 8 9 12 14
3 6 8 9 13 14	3 6 8 10 11 12	3 6 8 10 11 13
3 6 8 10 12 14	3 6 8 10 13 14	

c) with cardinality 7 (128 items)

1 3 4 7 9 11 12	1 3 4 7 9 11 13	1 3 4 7 9 12 14
1 3 4 7 9 13 14	1 3 4 7 10 11 12	1 3 4 7 10 11 13
1 3 4 7 10 12 14	1 3 4 7 10 13 14	1 3 4 8 9 11 12
1 3 4 8 9 11 13	1 3 4 8 9 12 14	1 3 4 8 9 13 14
1 3 4 8 10 11 12	1 3 4 8 10 11 13	1 3 4 8 10 12 14
1 3 4 8 10 13 14	1 3 5 7 9 11 12	1 3 5 7 9 11 13
1 3 5 7 9 12 14	1 3 5 7 9 13 14	1 3 5 7 10 11 12
1 3 5 7 10 11 13	1 3 5 7 10 12 14	1 3 5 7 10 13 14

1	3	5	8	9	11	12
1	3	5	8	9	13	14
1	3	5	8	10	12	14
1	4	6	7	9	11	13
1	4	6	7	10	11	12
1	4	6	7	10	13	14
1	4	6	8	9	12	14
1	4	6	8	10	11	13
1	5	6	7	9	11	12
1	5	6	7	9	13	14
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